Some multiplication properties of M2x2(F)

Dr. Najmaddin Hama Gareb.

Department of Mathematics, School of Science Education, University of Sulaimani, Kurdistan Region/ Iraq.

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Abstract

This work is divided into two parts; first we find all matrices which commutative with any given matrix, and second devoted to some multiplication commutative properties of M_{2x2}(F), where F is a field. Moreover some cases which the ring M_{2x2}(F) become commutative are studied.

Keywords: Multiplication commutative of two by two matrices over a field F.

Introduction

From [2] and [3] we recall that:
A non-empty set R with any two operations for simplicity we denoted them by (+) and (.) is called a ring if (R, +) is an abelian group, (R, .) is a semi-group and the multiplication (.) is distributed on the addition (+) from both sides, i.e.
a.(b+c) = a.b+a.c, and (b+c).a = b.a+c.a.
For simplicity also we write R is a ring instead of writing (R, +, .) is a ring and we write ab instead a.b. Further a ring R is called a commutative if ab = ba for all a and b in R, and a ring R is called has unity or identity 1 if and only if there exists an element say 1 in R such that 1.a = a = a.1, for all a in R.
(F, +, .) is a field if (F, +) and (F-{0}, .) are abelean groups, and the multiplication (.) is distributed on the addition (+) from both sides.
If Z is the set of all integer numbers, Q is the set of all rational numbers, ℝ is the set of all real numbers and C is the set of all complex numbers then:
(Z, +, .), (Q, +, .), (ℝ, +, .), and (C, +, .) are all examples on commutative rings with 1 as an identity, and (Q, +, .), (ℝ, +, .), (C, +, .) and (Z_n, +, .), where n is a prime number, are all examples on fields.

Further the set of all 2x2 matrices where elements in each matrix were taken from field F (we denote this set by M_{2x2}(F) with usual addition (+) and multiplication (.) on matrices forms a ring with identity $I_2 = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$, where e is the multiplicative identity of field F but (M_{2x2}(F), +, .) is not commutative ring since multiplication of matrices is not commutative, in general (see [1]), for example:

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Moreover commutative matrices, and theorems on commutative matrices have been studied by many authors (see [4], [5], [6], and [7]), and this idea leads us to introduce the following concept:

§1 Matrices which commutative with any given matrix.

The notion of commuting matrices was introduced by Cayley in his memoir on the theory of matrices, which also provided the first axiomatization of matrices. The first significant results proved on them were the above result of Frobenius in 1878.

It`s known that:

- The unit matrix commutes with all matrices.
- Diagonal matrices commute.
- If the product of two symmetric matrices is symmetric, then they must commute.
- The property of two matrices commuting is not transitive: A matrix A may commute with both B and C, and still B and C do not commute with each other. As an example, the unit matrix commutes with all matrices, which between them not all commute. If the set of matrices considered is restricted to Hermitian matrices without multiple eigenvalues, then commutativity is transitive, as a consequence of the characterization in terms of eigenvectors.

Now in this section we state and prove more results on multiplication commutative in the rings M_{2x2}(F) and we will begin by the following proposition in which, under certain condition (namely $0 \neq$ c), we find all matrices of the form $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ whose commutative with a given matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

Proposition 1.1

If a, b, c and d are elements in a field F and $0 \neq$ c, then the given matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is commutative with matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, if and only if $x = s(a-d)c^{-1} + r,\ y = sbc^{-1},\ z = s$ and $w = r$, where r and s are any elements in a field F.

Proof

Let A and B be two commutative matrices, i.e. AB= BA, so that

$$\begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix} = \begin{bmatrix} xa+yc & xb+yd \\ za+wc & zb+wd \end{bmatrix},$$

but this gives the following four equations:

$ax + bz = xa + yc$

$ay + bw = xb + yd$

$cx + dz = za + wc$

$cy + dw = zb + wd$

Or

$bx - fy - bw = 0$

$-cx + fz + cw = 0$

$cy - bz = 0$

$-cy + bz = 0.$

By solving the above homogeneous linear system which is consists of four equations in four unknowns namely x, y, z and w we get:

$x = s(a-d)c^{-1} + r$, $y = sbc^{-1}$, $z = s$ and $w = r$ for all s and r in F.

Therefore the matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is commute with matrix $B = \begin{bmatrix} s(a-d)c^{-1}+r & sbc^{-1} \\ s & r \end{bmatrix}$, where r and s are any elements in a field F.

Conversely in a matrix B let $w = r$, $z = s$, $y = (b/c)s$, and $x = \left[\frac{a-d}{c}\right] s + r$ for all s and r in F.

Then AB= $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s(a-d)c^{-1}+r & sbc^{-1} \\ s & r \end{bmatrix}$

$$= \begin{bmatrix} [(a-d)asc^{-1} + ar + bs] & [absc^{-1} + br] \\ as + rc & bs + rd \end{bmatrix}$$

= BA.

Remark 1

In the Proposition 1.1, of course c^{-1} exists since we assumed 0≠c and c in a field F.

Example 1.2

By given various values for arbitrary elements r and s in a field of real numbers ℝ in Proposition 1.1 we can find infinite number of matrices which they are commute with matrix A = $\begin{bmatrix} -2 & 5 \\ 3 & 7 \end{bmatrix}$ in $M_{2x2}(\mathbb{R})$, for example

B1 = $\begin{bmatrix} -2 & 5/3 \\ 1 & 1 \end{bmatrix}$, B2 = $\begin{bmatrix} -5 & 10/3 \\ 2 & 1 \end{bmatrix}$, B3 = $\begin{bmatrix} -1 & 5/3 \\ 1 & 2 \end{bmatrix}$, B4 = $\begin{bmatrix} -4 & 10/3 \\ 2 & 2 \end{bmatrix}$, . . . , etc.

Are all commute with A. But in a finite field we get a finite number of matrices which commute with a given matrix as in the following example.

Example 1.3

The matrix A = $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ in $M_{2x2}(Z_2)$ is commute with all of the following matrices:

$B1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Remark 2

All examples (like Example 1.3) can be calculated by converting Proposition 1.1 to mat lab program. For example in $M_{2x2}(Z_3)$, the following program which is writing by mat lab can find all matrices B_i which is commute with given matrix A= $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and the verifications $AB_i = B_iA$ are also appear for all values of i.

```
A=[0 1;1 1]
% m means mod
m=3
for r=0:m-1
  for s=0:m-1
    fprintf('If r=%g and s=%g then\n', [r s])
x=mod(((A(1, 1)-A(2, 2))/A(2, 1))*s +r,
m);
y= mod((A(1, 2)/A(2, 1))*s, m);
B=[x y;s r]
AB=mod(A*B, m)
BA=mod(B*A, m)
  end, end
A = 0        1
     1        1
m = 3
If r=0 and s=0 then
```

B =0 0
0 0
AB =
0 0
0 0
BA =
0 0
0 0
If r=0 and s=1 then
B =
2 1
1 0
AB =
1 0
0 1
BA =
1 0
0 1
If r=0 and s=2 then
B =
1 2
2 0
AB =
2 0
0 2
BA =
2 0
0 2
If r=1 and s=0 then
B =
1 0
0 1
AB =
0 1
1 1
BA =
0 1
1 1
If r=1 and s=1 then
B =
0 1
1 1
AB =
1 1
1 2
BA =
1 1
1 2
If r=1 and s=2 then
B =
2 2
2 1
AB =
2 1
1 0
BA =
2 1
1 0
If r=2 and s=0 then
B =
2 0
0 2
AB =
0 2
2 2
BA =
0 2
2 2
If r=2 and s=1 then
B =
1 1
1 2
AB =
1 2
2 0
BA =
1 2
2 0
If r=2 and s=2 then
B =

0	2
2	2

AB =

2	2
2	1

BA =

2	2
2	1

Now we will discuss the case c=0 in the Proposition 1.1 as follows:

Corollary 1.4

If a, b and d are elements in a field F, then the given matrix A = $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is commute with matrix $B = \begin{bmatrix} x & y \\ 0 & w \end{bmatrix}$ if and only if $x = r$, $y = b(r-s)(a-d)^{-1}$ and $w = s$, where r and s are any elements in a field F.

Proof

A and B are two commute matrices if and only if

$\begin{bmatrix} ax & ay+bw \\ 0 & dw \end{bmatrix}$ = $\begin{bmatrix} xa & xb+yd \\ 0 & wd \end{bmatrix}$ or if and only if

$ay + bw$ = $xb + yd$, but we know that one equation can find only one unknown, so we must assume that $x = r$, $w = s$ and consequently we can find $y = b(r-s)(a-d)^{-1}$, where r and s are any elements in a field F.

Remark 3

In the Corollary 1.4, of course $(a-d)^{-1}$ exists in a field F if and only if $a - d \neq 0$ and this occur when $a \neq d$.

Now in the following corollary we discuss the case $a = d$ of Corollary 1.4.

Corollary 1.5

If a and b are elements in a field F and b≠0, then the given matrix A = $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is commute with matrix $B = \begin{bmatrix} x & y \\ 0 & w \end{bmatrix}$ if and only if $x = r = w$, $y = s$, where r and s are any elements in a field F.

Proof

A and B are two commutative matrices if and only if

$\begin{bmatrix} ax & ay+bw \\ 0 & aw \end{bmatrix}$ = $\begin{bmatrix} xa & xb+ya \\ 0 & wa \end{bmatrix}$, or if and only if:

$ay + bw$ = $xb + ya$, but we know that from one equation we can find only one unknown, so we must assume that $x = r$ and $y = s$, where r and s are any elements in a field F. Moreover since b$\neq$ 0 hence b^{-1} exists in a field F and from equation bw = xb we can find $w = x = r$.

Remark 4

If b=0 then A becomes a diagonal matrix which is trivially commute with all matrices of the same size.

Example 1.6

The matrix A = $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ in $M_{2x2}(Z_3)$, is commute with all of the following matrices:

$B1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B2 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, B3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B4 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$. $B5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B6 =$

$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $B7 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$, $B8 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, and $B9 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Remark 5

Example 1.6 also can be calculated by converting Corollary 1.4 to the following short mat lab program:

```
A=[1 2;0 2]
% m means mod
m=3
for r=0:m-1
  for s=0:m-1
    fprintf('If r=%g and s=%g then\n', [r s])
y= mod(A(1, 2)*(r-s)/(A(1, 1)-A(2, 2)), m);
B=[r y;0 s]
AB=mod(A*B, m)
BA=mod(B*A, m)
  end, end
```

§2 Some multiplication commutative properties of M_{2x2}(F).

Here all matrices are in M_{2x2}(F), however our work is true in M_{nxn}(F) for all fixed positive integer n greater than or equal to two. In this section we study some of multiplication commutativity properties of such matrices.

Proposition 2.1

If a matrix A commute with matrices B and C then A is commute with the product and sum of them.

Proof

First we show that A is commute with the product of B and C as follows:

A(BC) = (AB)C, (by associative law)

= (BA)C, (by commutative law)

= B(AC), (by associative law)

= B(CA), (by commutative law)

= (BC)A, (by associative law)

Next we show that A is commute with the sum of B and C as follows:

A(B+C) = AB+AC, (by distributive law)

= BA+CA, (by commutative law)

= (B+C) A, (by distributive law).

Remark 6

The second part of Proposition 2.1 is also true if we write (-) instead of (+).

Corollary 2.2

For every natural number n If a matrix A commute with matrices B_1, B_2, ..., B_n then

(1) A($B_1.B_2 \ldots B_n$) =($B_1.B_2 \ldots B_n$) A,
(2) A($B_1+B_2+\ldots+B_n$)=($B_1+B_2+\ldots+B_n$)A.

Proof

We use mathematical induction to proof this Proposition as follows:

For n=1 and 2 the rules are true by operative part of Corollary and Proposition 2.1.

Next we assume that the rules are true for n=k, then by using this assumption, associative law and distributive law we prove the rules are true for n= k+1 as follows:

(1) A($B_1.B_2 \ldots B_{k+1}$)
= A[($B_1.B_2 \ldots B_k$) B_{k+1}]
= [A ($B_1.B_2 \ldots B_k$)] B_{k+1}
= [($B_1.B_2 \ldots B_k$)A] B_{k+1}
= ($B_1.B_2 \ldots B_k$) [AB_{k+1}]
= ($B_1.B_2 \ldots B_k$) [B_{k+1}A]

$= [(B_1.B_2 \ldots B_k)\, B_{k+1}]$A
$= (B_1.B_2 \ldots B_{k+1})$ A.

(2) A$(B_1+B_2+\ldots+B_{k+1})$
$=$ A$[(B_1+B_2+\ldots+B_k)+B_{k+1}]$
$=$ A$(B_1+B_2+\ldots+B_k)+$AB_{k+1}
$= (B_1+B_2+\ldots+B_k)$A$+B_{k+1}$A
$= [(B_1+B_2+\ldots+B_k)+B_{k+1}]$A
$= (B_1+B_2+\ldots+B_{k+1})$A.

Therefore the rules are true for every natural number n.

Corollary 2.3

Let A be a fixed matrix in a ring (M_{nxn}(F), +, .) and S(A) ={B∈ M_{nxn}(F); AB=BA}, then (S(A), +, .) is a sub-ring of the ring (M_{nxn}(F), +, .).

Proof

Since there exists the matrix A in M_{nxn}(F) and $AA = AA$ (i.e. A commute with itself) , so $A \in S$(A) and S(A) $\neq \emptyset$. Further from definition of S(A) it is clear that S(A) is the subset of M_{nxn}(F), thus;

$\emptyset \neq$ S(A) $\subseteq M_{nxn}$(F).

Let B and C be any two elements in S(A), then by Proposition 2.1, we can say that

A(B-C) =(B-C)A, and A(BC) = (BC) A, hence B-C and BC are also in S(A). Therefore (S(A), +, .) is a sub-ring of the ring (M_{nxn}(F), +, .).

Proposition 2.4

If a matrix A commute with a nonsingular matrix B then A is commute with B^{-1}.

Proof

A$B^{-1} = I_n$(AB^{-1})

$= (B^{-1}$B)(AB^{-1})

$= B^{-1}$ (BA) B^{-1}

$= B^{-1}$ (AB) B^{-1}

$= (B^{-1}$A)(BB^{-1})

$= (B^{-1}$A). I_n

$= B^{-1}$A.

Corollary 2.5

If a matrix A commute with the nonsingular matrices $B_1, B_2, \ldots, B_n$ then A is commute with $B_1^{-1}.\, B_2^{-1}. \ldots B_n^{-1}$.

Proof

We use mathematical induction to proof this Proposition as follows:

For n=1 the rule is true by Proposition 2.4.

Next we assume that the rule is true for n=k, then by using this assumption and associative law we prove the rule is true for n= k+1 as follows:

A$(B_1^{-1}.\, B_2^{-1}. \ldots B_{k+1}^{-1})$
$=$ A$[(B_1^{-1}.\, B_2^{-1}. \ldots B_k^{-1})\, B_{k+1}^{-1}]$
$= [$A$(B_1^{-1}.\, B_2^{-1}. \ldots B_k^{-1})]\, B_{k+1}^{-1}$
$= [(B_1^{-1}.\, B_2^{-1}. \ldots B_k^{-1})A]\, B_{k+1}^{-1}$
$= (B_1^{-1}.\, B_2^{-1}. \ldots B_k^{-1})[$A $B_{k+1}^{-1}]$
$= (B_1^{-1}.\, B_2^{-1}. \ldots B_k^{-1})[B_{k+1}^{-1}A]$
$= [(B_1^{-1}.\, B_2^{-1}. \ldots B_k^{-1})\, B_{k+1}^{-1}]$A
$= (B_1^{-1}.\, B_2^{-1}. \ldots B_{k+1}^{-1})$A.

Therefore the rule is true for every natural number n.

Proposition 2.6

For all A_i in M_{nxn}(F), if S(A_i) ={B∈ M_{nxn}(F); A_iB=BA_i}, then M_{nxn}(F) is commutative ring if and only if the intersection of all S(A_i) is equal to M_{nxn}(F).

Proof

Let M_{nxn}(F) be a commutative ring, then by Corollary 2.3, for all A_i in M_{nxn}(F), we have S(A_i) is a sub-ring of M_{nxn} (F) and consequently $\cap_{A_i \in \text{Mnxn(F)}}$ S(A_i) is a sub-ring and to show M_{nxn} (F) is a subset of $\cap_{A_i \in \text{Mnxn(F)}}$ S(A_i) , let X∈ M_{nxn}(F), which is commutative ring, then XA_i= A_iX, for all A_i in M_{nxn} (F), hence X ∈ S(A_i), for all A_i ∈ M_{nxn} (F), thus X ∈$\cap_{A_i \in \text{Mnxn(F)}}$ S(A_i) , therefore $\cap_{A_i \in \text{Mnxn(F)}}$ S(A_i) = M_{nxn}(F).

Conversely, let $\cap_{A_i \in \text{Mnxn(F)}}$ S(A_i) = M_{nxn}(F), then to show M_{nxn}(F) is commutative ring let X and Y be any two matrices in M_{nxn}(F) = $\cap_{A_i \in \text{Mnxn(F)}}$ S(A_i) then X and Y ∈ S(X) and S(Y), hence XY =YX.

Example 2.7

If R = {$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$; a, b∈ Z_2}, then (R, +, .) is a commutative ring and S(A_i) = R, for all A_i in R, hence $\cap_{A_i \in \text{R}}$ S(A_i) = R.

Example 2.8

Clearly if R = {$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$; a, b, c∈ Z_2}, then R is not commutative and $\cap_{A_i \in \text{R}}$ S(A_i) ≠ R.

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